Internal thermal noise in the LIGO test masses: A direct approach

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Really big interferometer

Figure: https://www.britannica.com/topic/Laser-Interferometer-Gravitational-wave-Observatory/media/1/1562918/205865

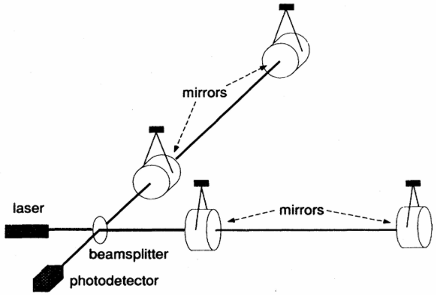
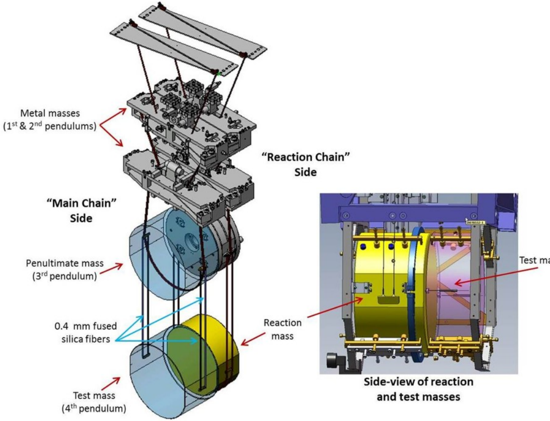


FIG. 1. Schematic view of a LIGO interferometer.

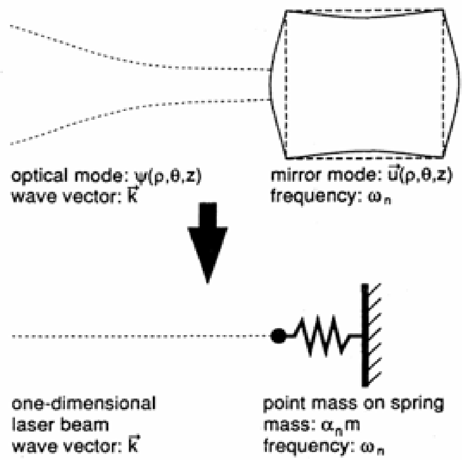
Figure: A. Gillespie and F. Raab (1994)

ultra high vacuum



quadruple-pendulum system

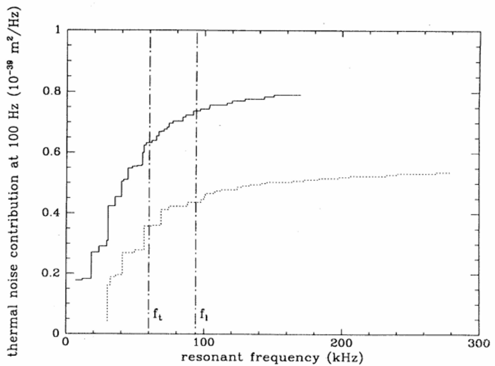
Thermal Noise using Normal-Mode Expansion



$$a = \frac{U_n}{\frac{1}{2} m w_n^2 \Delta l_n^2}$$

Normal-Mode Expansion

$$S_x(f) \approx \sum_n \frac{4k_bT}{a_n m w_n^2} \frac{\varphi_n(w)}{w}$$



Issues with Normal-Mode Expansion

1. Assumption that different normal modes have independent Langevin forces
2. For a small laser beam diameter, the sum over normal modes converges very slowly

Use of generalized fluctuation-dissipation theorem

Consider a beam interacting with test mass Measurement of fluctuation of x

$$x(t) = \int f(\vec{r}) y(\vec{r}, t) d^2 r$$

where f is the distribution of the beam intensity

$$\int f(r) d^2\vec{r} = 1$$

Spectral density of generalized fluctuation-dissipation theorem

$$S_x(f) = \frac{k_b T}{\pi^2 f^2} |Re[Y(f)]|$$

Use of generalized fluctuation-dissipation theorem

driving force F(t) and as the interaction term in the Hamiltonian

$$H_{int} = -F(t)x$$

$$H_{int} = -\int P(\vec{r})y(\vec{r},t)d^2r$$

where P is the pressure of F over impact area

$$P(\vec{r},t) = F(t)f(\vec{r})$$

Use of generalized fluctuation-dissipation theorem

suppose we apply a oscillating pressure on the rest mass

$$P(\vec{r}, t) = F_0 cos(2\pi ft) f(\vec{r})$$

Then the admittance is then given as

$$Y(f) = 2\pi f x(f)/F(f)$$

where the real component is given by

$$|Re[Y(f)| = \frac{2W_{diss}}{F_0^2}$$

Spectral density of test mass

$$S_x(f) = \frac{2k_bT}{\pi^2 f^2}\frac{W_{diss}}{F_0^2}$$

1. Apply oscillating pressure
2. determine the work dissipated
3. derive the spectral density

Thermal noise due to homogeneously distributed damping

friction is conventionalized by an imaginary part of the material's Young's modulus

$$E = E_0[1 + \phi(f)]$$

Power Dissipated is

$$W_{diss} = 2\pi f U_{max} \phi(f)$$

U is the energy of elastic deformation of the rest mass

$$U_{max} = \frac{F_0^2}{\pi^2 E_0 r_0}(1 - \sigma^2) I[1 + O(\frac{r_0}{R})]$$

APPENDIX: THE STRAIN ENERGY IN A TEST MASS SUBJECTED TO A GAUSSIAN DISTRIBUTED SURFACE PRESSURE

The objective of this appendix is to derive Eq. (14) of Sec. III for the energy of elastic strain in a cylindrical test mass when the pressure $P(\vec{r}) = F_0 f(\vec{r})$ is applied to one of its circular faces. (As was discussed in Sec. III, we can assume that the pressure is constant in time since LIGO's detection frequencies are much lower than the lowest normal-mode frequency.) For a circular laser beam with a Gaussian intensity profile $f(\vec{r})$ is given by [cf. Eq. (13)]

$$f(\vec{r}) = \frac{1}{\pi r_0^2} e^{-r^2/r_0^2}, \tag{A1}$$

where we assume that the center of the light spot coincides with the center of the test-mass circular face.

If the radius of the laser beam r_0 is small compared to the size of the test mass, we can approximate the test mass by an infinite elastic half-space. Then our calculation of the elastic energy is correct up to a fractional accuracy of $O(r_0/R)$, where R is the characteristic size of the test mass.

Let $y(\vec{r})$ be the normal displacement of the surface at location $\vec{r}$ under the action of the pressure $P(\vec{r})$. In the linear approximation of small strains,

$$y(\vec{r}) = \int G(\vec{r}, \vec{r}') P(\vec{r}') d^2 r', \tag{A2}$$

where $G(\vec{r}, \vec{r}')$ is a Green's function. The calculation of G is a nontrivial albeit standard exercise in elasticity theory [10], which gives

$$G(\vec{r}, \vec{r}') = \frac{1-\sigma^2}{\pi E_0} \frac{1}{|\vec{r} - \vec{r}'|}, \tag{A3}$$

where σ is the Poisson ratio and E_0 the Young's modulus of the material. The elastic energy stored in the material is

$$U_{\max} = \frac{1}{2} \int P(\vec{r}) y(\vec{r}) d^2 r = \frac{1}{2} \frac{1-\sigma^2}{\pi E_0} \int \frac{P(\vec{r}) P(\vec{r}')}{|\vec{r} - \vec{r}'|} d^2 r d^2 r'$$
$$= \frac{1}{2} \frac{1-\sigma^2}{\pi^3 E_0 r_0^4} F_0^2 \int \frac{e^{-(r^2 + r'^2)/r_0^2}}{\sqrt{r^2 + r'^2 - 2 r r' \cos\theta}} d^2 r d^2 r', \tag{A4}$$

where θ is the angle between $\vec{r}$ and $\vec{r}'$. The integral in the last term of Eq. (A4) (as was pointed out by Glenn Sobermann) can be taken by introducing "polar" coordinates R and ϕ: $r = R\cos\phi$, $r' = R\sin\phi$. One then integrates out the radial part of the integrand and expands the remaining angular part in a power series with respect to $\cos\theta$; termwise integration of this power series finally yields Eq. (14) [up to a fractional error of $O(r_0/R)$]

$$U_{\max} \simeq \frac{F_0^2}{\pi^2 E_0 r_0} (1-\sigma^2) I, \tag{A5}$$

where

$$I = \frac{\pi^{3/2}}{4} \left[1 + \sum_{n=1}^{\infty} \frac{(4n-1)!!}{(2n)! 4^n (2n+1)} \right] \simeq 1.873\,22. \tag{A6}$$

It can be shown that if, instead of an infinite half-space, we consider a finite cylindrical test mass, the leading fractional correction to the elastic energy is of the order $O(r_0/R)$.

Thermal noise due to homogeneously distributed damping

Gives expression for spectral density

$$S_x(f) = \frac{4k_bT}{f}\frac{(1-\sigma^2)}{\pi^3 E_0 r_0} I\phi[1 + O(\frac{r_0}{R})]$$

Comparing estimation

1. using Normal Mode with 30 Modes $S_x^{GR}(100Hz) \simeq 8.0 * 10^-40 m^2/Hz$
2. using generalized fluctuation dissipation theorem $S_x(100Hz) \simeq 8.7 * 10^-40 m^2/Hz$
3. software aided numerical computation of U $S_x(100Hz) \simeq 8.76 * 10^-40 m^2/Hz$

Improvement over normal modes method

1. Becomes more exact when the laser diameter is small
2. Less computationally intensive
3. Simpler process

Case of surface damping

The power dissipated at each point of the material is proportional to the square of the stress at t

$$W^{coatingdiss} \propto (\frac{F_0}{r_0^2})^2 r_0^2 = \frac{F_0^2}{r_0^2}$$

$$S_x(Boundary) \propto \frac{1}{r_0^2}$$

$$S_x(bulk) \propto \frac{1}{r_0}$$

one dimensional illustration

Improvement over normal modes method

1. Becomes more exact when the laser diameter is small
2. Less computationally intensive
3. Simpler process
4. Accounts for surface of beam imperfections better

Sources

1. LIGO Lab — Caltech — MIT. (n.d.). LIGO Lab — Caltech. https://www.ligo.caltech.edu/
2. A.GillespieandF.Raab,Phys.Rev.D52,577 1995
3. Levin, Y. (1998). Internal thermal noise in the LIGO test masses: A direct approach. Physical Review. D, Particles and Fields, 57(2), 659–663. https://doi.org/10.1103/PhysRevD.57.659